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BIASED POSITIONAL GAMES FOR WHICH RANDOM STRATEGIES ARE NEARLY OPTIMAL

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For a graph G and natural numbers n and q let $\mathbf{G}(G;n,q)$ be the game on the complete graph K_n in which two players, Maker and Breaker, alternately claim 1 and q edges respectively. Maker's aim is to build a copy of G while Breaker tries to prevent it. Let $m(G)=\max\left\{\frac{e(H)-1}{v(H)-2}:\ H\subseteq G, v(H)\geq 3\right\}$. It is shown that there exist constants c_0 and C_0 such that Maker has a winning strategy in $\mathbf{G}(G;n,q)$ if $q\le c_0 n^{1/m(G)}$, while for $q\ge C_0 n^{1/m(G)}$ the game can be won by Breaker.

1. Introduction

Let n, q be natural numbers and G be a graph. In the note we study a game $\mathbf{G}(G;n,q)$ played on the complete graph K_n on n vertices by two players, Maker and Breaker, who build two edge-disjoint subgraphs of K_n. In each round of the game Maker chooses an edge of K_n, which has not been claimed previously, and Breaker answers by picking at most q new edges from K_n. The game ends if each of $\binom{n}{2}$ edges of K_n is claimed by either of the players. If the graph constructed during the game by Maker contains a copy of G he wins, otherwise he loses.

The game $\mathbf{G}(G;n,q)$ is a special case of positional games on graphs and hypergraphs extensively studied by Beck [1–4]. Most of the results on $\mathbf{G}(G;n,q)$ concern graphs G whose size depends on n, e.g. hamiltonian cycles, spanning trees, big stars or large complete subgraphs. For example,

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for $q=1$, the size of the biggest clique Maker can build in K_n is of the order $\log n$ (see [1]). A fast winning strategy for Maker in $\mathbf{G}(K_m;n,1)$ with $m=(1+o(1))\log_2 n$ was found by Pekeč [8], who also showed that a similar strategy can by applied to win $\mathbf{G}(K_m;n,q)$ for an appropriately chosen $m=m(n,q)$. His result and a straightforward application of Beck's theorem [2] imply that the maximum m for which Maker can win $\mathbf{G}(K_m;n,q)$ is of the order $\log n/\log q$, provided $q=q(n)$ and $1/q=o(1)$.

Chvátal and Erdős [5] asked about the "threshold value" $q_0=q_0(n)$ such that for $q \le q_0(n)$ Maker has a winning strategy for $\mathbf{G}(G;n,q)$ while for $q > q_0(n)$ the game can be won by Breaker. It is known that $q_0(n)$ is of order $n/\log n$ if Maker wants to build a spanning tree [5] or a hamiltonian cycle [3].

Beck pointed out that there are remarkable similarities between results on positional games played on graphs and threshold properties of random graphs (see [2, 4]). In this note we explore this relation and use results on random graphs to show that in $\mathbf{G}(G;n,q)$ the "random strategy" is almost optimal for Maker. We consider only the case in which G is fixed, i.e. it does not depend on n. This kind of games, when G is a clique, was proposed by Chvátal and Erdős [5], who proved that the game $\mathbf{G}(K_3;n,q)$ can be won by Maker if $q<(2n+2)^{1/2}-5/2$, and by Breaker if $q\ge 2n^{1/2}$.

For a graph H let $v(H)$ denote the number of vertices of H and $e(H)$ stand for the number of its edges. For a graph G with at least three vertices we define

$$m(G) = \max_{\substack{H\subseteq G\\ v(H)\ge 3}} \frac{e(H)-1}{v(H)-2}.$$

The main result of this note states that the function $q=q(n)$, "critical" for $\mathbf{G}(G;n,q)$, is of the order $n^{1/m(G)}$.

Theorem 1. *For every graph G which contains at least 3 nonisolated vertices there exist positive constants c_0, C_0 and n_0 such that for every $n\ge n_0$ the following holds.*

(i) *If $q \le c_0 n^{1/m(G)}$ then Maker has a winning strategy in the game $\mathbf{G}(G;n,q)$.*
(ii) *If $q\ge C_0 n^{1/m(G)}$ then Breaker can win $\mathbf{G}(G;n,q)$.*

The probabilistic proof of the first part of Theorem 1 is given in the next section. Then we apply a "derandomization" argument, similar to that used by Erdős and Selfridge [6] and Beck [2], to show the second part of the assertion.

2. Maker's strategy—a probabilistic approach

In this section we show that if $q \le c_0 n^{1/m(G)}$ and the constant $c_0 > 0$ is small enough, then Maker can win $\mathbf{G}(G;n,q)$. Let us consider first the case in which G is a forest. If G consists of independent edges then $m(G)=1/2$ and one can easily see that Maker can win in $e(G)$ moves as long as $q \le \binom{n}{2}/(e(G)-1)-2n$. On the other hand if G contains a path on three vertices then $m(G)=1$ and, clearly, for $q \le (n-e(G))/(e(G)-1)$ it takes Maker's $e(G)$ moves to build G.

Assume now that G contains a cycle. Let $\overline{\mathbf{G}}(G;n,q)$ denote the modification of $\mathbf{G}(G;n,q)$ in which Breaker has all the information about Maker's moves, but Maker cannot see the moves of his opponent. Thus, if Maker chooses a pair of vertices $\{v,w\}$ of K_n, it might happen that $\{v,w\}$ has been previously claimed by Breaker. In such a case the pair $\{v,w\}$ is marked as a *failure* and Maker loses his move. We say that Maker plays according to the *random strategy*, if in each move he selects a pair of vertices chosen uniformly at random among all pairs which has not been claimed by him so far. Note that if Breaker has a winning strategy for $\mathbf{G}(G;n,q)$ he can apply it to win $\overline{\mathbf{G}}(G;n,q)$. Thus, in order to prove Theorem 1(i), it is enough to show that if Maker plays according to the random strategy, there is a positive probability that he wins $\overline{\mathbf{G}}(G;n,q)$.

Theorem 2. *For every G which contains a cycle there exist constants $c_0 > 0$ and n_0 such that for every $n \ge n_0$ and $q \le c_0 n^{1/m(G)}$ Maker has a random strategy in $\overline{\mathbf{G}}(G;n,q)$ which succeeds with probability at least $1/3$ against any strategy of Breaker.*

Remark. In fact we show that for $q \le c_0 n^{1/m(G)}$ Maker's random strategy succeeds with probability tending to 1 as $n \to \infty$.

The proof of Theorem 2 relies on certain properties of random graphs. Let us recall that $\mathbb{G}(n,M)$ is a graph chosen uniformly at random from the family of all subgraphs of K_n with n vertices and M edges, while $\mathbb{G}(n,p)$ denotes the graph obtained by removing edges of K_n with probability $1-p$, independently for each edge. Typically, we are interested in the behaviour of the random graphs $\mathbb{G}(n,M)$ and $\mathbb{G}(n,p)$ for large values of n, where parameters M and p vary as functions of n.

Our argument is based on the following upper bound of the probability that a random graph contains no copies of a given graph G proved by Janson, Łuczak and Ruciński [7].

Lemma 3. *For every graph G containing a cycle there exist constants $c_1 > 0$ and n_1 such that for every $n \ge n_1$ and $n^{-1/m(G)} \le p \le 3n^{-1/m(G)}$*

$$\mathbb{P}(\mathbb{G}(n,p) \not\supseteq G) \le \exp(-c_1 n^2 p)\,. \qquad \blacksquare$$

Using elementary properties of the binomial distribution one can deduce from Lemma 3 its $\mathbb{G}(n,M)$ counterpart.

Lemma 3′. *For every graph G with at least one cycle there exist constants $c_1'>0$ and n_1' such that for every $n\ge n_1'$ and $\bar M=\lfloor n^{2-1/m(G)}\rfloor$ we have*

$$\mathbb{P}(\mathbb{G}(n,\bar M)\not\supseteq G)\le \exp(-c_1'\bar M)\,. \qquad \blacksquare$$

In our argument we shall need only the following consequence of the above result.

Lemma 4. *For every graph G containing at least one cycle there exist constants $0<\delta<1$ and n_2 such that for $n\ge n_2$ and $M=2\lfloor n^{2-1/m(G)}\rfloor$ with probability at least $2/3$ each subgraph of $\mathbb{G}(n,M)$ with $\lfloor(1-\delta)M\rfloor$ edges contains a copy of G.*

Proof. Let $n_2=n_1'$ and $0<\delta<1/2$ be a small constant such that

$$\delta-\delta\log\delta<c_1'/3\,,$$

where n_1' and c_1' are such that the assertion of Lemma 3′ holds. Consider pairs (F,F'), where F is a subgraph of K_n with M edges, and F' is a subgraph of F with $\lfloor(1-\delta)M\rfloor$ edges which contains no copies of G. Using Lemma 3′ (with $\bar M=M/2$) to estimate the number of choices for F', one can bound from above the number of such pairs by

$$\exp(-c_1'M/2)\binom{\binom{n}{2}}{(1-\delta)M}\binom{\binom{n}{2}-(1-\delta)M}{\delta M}\,.$$

Thus, the number of choices for F is bounded from above by

$$\exp(-c_1'M/2)\binom{M}{\delta M}\binom{\binom{n}{2}}{M}\le\left(\frac{e}{\delta}\right)^{\delta M}\exp(-c_1'M/2)\binom{\binom{n}{2}}{M}\le\frac{1}{3}\binom{\binom{n}{2}}{M},$$

provided n is large enough. $\blacksquare$

Remark. Note that the assertion of Lemma 4 is equivalent to the statement that for some constant $\delta'>0$ and every $n\ge n_0$ with probability at least $2/3$ the random graph $\mathbb{G}(n,M)$ contains at least $\delta'M$ edge-disjoint copies of G. Since for a given graph H the expected number of copies of H in $\mathbb{G}(n,M)$ is $O(n^{v(H)-2e(H)}M^{e(H)})$, for any M for which the above statement holds we must have

$$\min_{H\subseteq G}\{n^{v(H)-2e(H)}M^{e(H)}\}=\Omega(M)\,.$$

In fact this is the condition which determines the choice of $M=M(n)$ in Lemma 4 (and thus $m(G)$ and $q=q(n)$).

Proof of Theorem 2. Let $q=0.1\delta n^{1/m(G)}$ and $n>n_2$, where $\delta>0$ and n_2 are chosen in such a way that the assertion of Lemma 4 holds. Consider the game $\overline{\mathbf{G}}(G;n,q)$ in which Maker plays the random strategy. Let us consider the first

$$M = 2\lfloor n^{2-1/m(G)} \rfloor \le \frac{\delta}{2(q+1)}\binom{n}{2}$$

rounds of the game. Clearly, the graph whose edge set consists of all pairs Maker has claimed up to this point can be viewed as $\mathbb{G}(n,M)$, although some of these pairs may be failures, i.e. they have been already claimed by Breaker. Nonetheless, during the first M rounds of the game both players have selected not more than $\delta/2$ of all possible pairs, and so, for $i=1,\dots,M$, the probability that a pair chosen by Maker in his ith move is a failure is bounded from above by $\delta/2$. Consequently, for large M, with probability at least $2/3$ at most δM of the pairs claimed by Maker are failures. Hence, due to Lemma 4, with probability at least $1/3$ the graph built by Maker in the first M moves contains a copy of G. Thus, Maker's random strategy succeeds with probability at least $1/3$. ∎

3. Breaker's strategy

In order to prove Theorem 1(ii) we shall need some results on positional games played on hypergraphs. Let us recall that a *hypergraph* $\mathcal{H}$ is a pair $(V(\mathcal{H}),E(\mathcal{H}))$, where $V(\mathcal{H})$ is the set of vertices of $\mathcal{H}$ and the set $E(\mathcal{H})$ of edges of $\mathcal{H}$ is a family of nonempty subsets of $V(\mathcal{H})$, We emphasize that, unlike in the graph case, we allow in $\mathcal{H}$ multiple edges i.e., a subset of $V(\mathcal{H})$ may appear as an edge of $\mathcal{H}$ several times.

Following Erdős and Selfridge [6] and Beck [2], we define $(\mathcal{H},p,q)$-game as the game in which Maker and Breaker alternately select respectively p and q previously unclaimed vertices of the hypergraph $\mathcal{H}$ until all the vertices have been claimed by either of the player. If at the end of the game Maker have chosen all vertices of some $A\in E(\mathcal{H})$, he wins, otherwise he loses.

For a given hypergraph $\mathcal{H}$ and natural numbers p,q let

$$f(\mathcal{H},p,q) = \sum_{A\in E(\mathcal{H})} (1+q)^{-|A|/p}.$$

Beck [2] showed the winning strategy for Breaker in the $(\mathcal{H},p,q)$-game if $f(\mathcal{H},p,q)<(1+q)^{-1}$. His argument relied on the fact that Breaker can play in such a way that the function

$$g(M,B) = \sum_{\substack{A \in E(\mathcal{H}) \\ A \cap B = \emptyset}} (1+q)^{-|A\setminus M|/p},$$

where M and B denote the sets of Maker's and Breaker's vertices respectively, never increases if it is evaluated after every move of Maker. Note that the game starts with $g(\emptyset,\emptyset) = f(\mathcal{H},p,q)$ and after the first Maker's move it can increase to at most $(q+1)f(\mathcal{H},p,q)$. Observe also that if at some stage of the game Maker's graph M contains $k \geq 0$ edges of $\mathcal{H}$, then $g(M,B) \geq k$. Thus, if in each of his moves Breaker tries to minimize the value of g, he can prevent Maker from claiming more than $(q+1)f(\mathcal{H},n,q)$ edges of $\mathcal{H}$ and the following holds.

Lemma 5. *In every $(\mathcal{H},p,q)$-game Breaker has a strategy such that at the end of the game at most $(1+q)f(\mathcal{H},p,q)$ edges of the hypergraph $\mathcal{H}$ have all their vertices claimed by Maker.* ∎

Now let us return to the game $\mathbf{G}(G;n,q)$. Suppose that $m(G) = \frac{e(H)-1}{v(H)-2}$ for some $H \subseteq G$ with $v(H) \geq 3$ and H is a minimal subgraph with this property. Clearly, if Breaker has a winning strategy in $\mathbf{G}(H;n,q)$ then, with the same strategy, he can also win $\mathbf{G}(G;n,q)$. Hence, it suffices to prove Theorem 1(ii) for graphs G which are *m-maximal*, i.e. which fulfil the condition

$$m(G) = \frac{e(G)-1}{v(G)-2} > m(H) \quad \text{for every } H \subset G,\ v(H) \geq 3.$$

For a m-maximal graph G we define a *$\bar{G}$-graph* $F^{v,w}$ as a graph F with two distinguished vertices v and w such that $F + \{v,w\}$ is a copy of G. Note that every m-maximal graph G which contains a cycle is 2-edge-connected, so for such a graph G each $\bar{G}$-graph is connected.

For a graph H we denote by $V(H)$ its vertex set and by $E(H)$ the set of its edges. Let $\{F_1^{v_1w_1},\dots,F_t^{v_tw_t}\}$ be a family of different $\bar{G}$-graphs. If $|\bigcap_{i=1}^t V(F_i^{v_iw_i})| \geq 2$ we call a family $\{F_1^{v_1w_1},\dots,F_t^{v_tw_t}\}$ a *t-fan*, if furthermore $|\bigcap_{i=1}^t V(F_i^{v_iw_i})| \geq 3$ we say that $\{F_1^{v_1w_1},\dots,F_t^{v_tw_t}\}$ forms a *t-flower*. A t-fan $\{F_1,\dots,F_t\}$ is *simple* if for $1 \leq i < j \leq t$ we have $|V(F_i^{v_iw_i}) \cap V(F_j^{v_jw_j})| = 2$. Finally, if at some moment of the game $\mathbf{G}(G;n,q)$ Maker's graph contains a $\bar{G}$-graph $F^{v,w}$ such that the pair $\{v,w\}$ has not been claimed by Breaker yet then we call $F^{v,w}$ *dangerous*. Similarly, we say a t-flower (or a t-fan) in Maker's graph is *dangerous* if it consists of t dangerous $\bar{G}$-graphs.

Lemma 6. *For every m-maximal graph G that contains a cycle there exist positive constants C_1, n_1 and $\delta < 1$ such that for every $n \geq n_1$ and $q \geq C_1 n^{1/m(G)}$ Breaker has a strategy such that at each moment of the game $\mathbf{G}(G;n,q)$ there are no dangerous s-flowers in Maker's graph, where $s = \lfloor q^{1-\delta} \rfloor$.*

Proof. Let G be a graph which fulfils the assumption of the lemma. If $v(G) = 3$ then there are no s-flowers with $s \geq 2$ and the assertion holds. Thus, assume that $v(G) > 3$ and set

$$\delta_1 = 1 - m(G)\Big(\max_{\substack{H \subset G \\ v(H) \geq 3}} \frac{e(G) - e(H)}{v(G) - v(H)} \Big)^{-1}.$$

Notice that $0 < \delta_1 < 1$ because G is m-maximal.

First we consider the game $\mathbf{G}(G;n,q_1)$ with $q_1 = \lceil q^{1-\delta_1/2} \rceil$ and show there exists a constant $t = t(G)$ such that Breaker in $\mathbf{G}(G;n,q_1)$ can prevent a t-*cluster*, i.e. a graph that consists of t copies of G which intersect on three (or more) vertices.

Breaker's strategy for $\mathbf{G}(G;n,q_1)$ will be based on his strategy stated in Lemma 5 for the $(\mathcal{H},1,q_1)$-game where $\mathcal{H}$ is the hypergraph with vertex set $E(K_n)$ and such that every set of the edges of a t-cluster forms an edge of $\mathcal{H}$ (note that a t-cluster is uniquely determined by the set of its edges). Thus, we need to estimate the value of $f(\mathcal{H},1,q_1)$. For every t-cluster let us order t copies $G_1, \ldots, G_t$ of G in such a way that for each $j = 1, \ldots, t_1$, we have $V(G_j) \not\subseteq \bigcup_{i=1}^{j-1} V(G_i)$, and t_1 is the largest number with this property. Then, clearly, $\frac{v-3}{v(G)-3} \leq t_1 \leq v - 3$, where $v = \left|\bigcup_{i=1}^{t} V(G_i)\right| = \left|\bigcup_{i=1}^{t_1} V(G_i)\right|$, and

$$f(\mathcal{H},1,q_1) = (q_1+1) \sum_{\substack{F \subset K_n \\ F \text{ is a } t\text{-cluster}}} (q_1+1)^{-e(F)}$$

$$\leq \binom{n}{3} [v(G)!]^t \sum_{\substack{v < tv(G) \\ \binom{v-3}{v(G)-3} v(G)! \geq t}} \sum_{t_1 = \lceil \frac{v-3}{v(G)-3} \rceil}^{v-3} \binom{v-3}{v(G)-3}^{t-t_1}$$

$$\times \Big(\sum_{\substack{H \subset G \\ 2 < v(H) < v(G)}} \binom{v-3}{v(H)-3} \binom{n-3}{v(G)-v(H)} (q_1+1)^{e(H)-e(G)} \Big)^{t_1}.$$

Indeed, the factor $\binom{n}{3}$ counts the possible choices for the vertices in the common intersection of $G_1, \ldots, G_t$, the sum in the paranthesis stands for the number of ways one can add G_j to the copies $G_1,\dots,G_{j-1}$ we have chosen so far (here by H we denote all possible intersections $E(G_j)\cap\bigcup_{i=1}^{j-1}E(G_i)$), and the first two sums give a crude upper bound for the number of choices of the last $t-t_1$ copies of G. Thus, if by H' we denote a subgraph of G which maximizes the terms of the interior sum, then for some constant c_1, which depends only on G,

$$f(\mathcal{H},1,q_1) \le \\ n^3 c_1^t \sum_{v=\lceil t^{1/v(G)}/3\rceil}^{tv(G)} \left(\frac{v}{v(G)}\right)^{v(G)t} \sum_{t_1=\lceil v/v(G)\rceil}^{v} \left(n^{v(G)-v(H')} q_1^{e(H')-e(G)}\right)^{t_1}.$$

Let $q\ge n^{1/m(G)}$. Then, by the choice of δ_1, we get

$$n^{v(G)-v(H')} q_1^{-e(G)+e(H')} \le (q^{1-\delta_1} q_1^{-1})^{e(G)-e(H')} \le q^{-\delta_1/2},$$

and so

$$f(\mathcal{H},1,q_1) < q^{3m(G)} c_1^t t^2 v(G)^2 t^{tv(G)} q^{-\delta_1 t'}$$

with $t'=t^{1/v(G)}/6v(G)$. Now set

$$t = \lceil 12(1+3m(G))v(G)\delta_1^{-1}\rceil^{v(G)},$$

so that

$$3m(G) - \delta_1 t' \le -1 - \delta_1 t'/2 .$$

Then the number of the edges of $\mathcal{H}$ which consist of Maker's vertices at the end of the $(\mathcal{H},1,q_1)$-game is bounded from above by

$$(q+1)f(\mathcal{H},1,q_1) \le c_1^t t^{tv(G)+2} v(G)^2 (q+1) q^{-1} q^{-\delta_1 t'/2} < 1,$$

provided q is big enough and Breaker plays his strategy as given in Lemma 5.

Thus Breaker in $\mathbf{G}(G;n,q_1)$ has a strategy which prevents Maker from building a t-cluster. Observe also that this strategy does not allow Maker to obtain a dangerous s-flower with $s\ge t(q_1+1)$. Indeed, suppose it is not the case and at some moment Maker succeeds in building such an s-flower $\{F_1^{v_1w_1},\dots,F_s^{v_sw_s}\}$. Then he can claim a pair which appears most frequently among $\{v_1,w_1\},\dots\{v_s,w_s\}$ and continue to do so until all the pairs $\{v_i,w_i\}$, $i=1,\dots,s$, have been claimed by either of the players. But in this way Maker could build at least $s/(q_1+1)\ge t$ copies of G intersecting on three vertices, which, as we have just shown, is impossible.

In order to complete the proof it is enough to observe that the above strategy for $\mathbf{G}(G;n,q_1)$ can be used by Breaker also in $\mathbf{G}(G;n,q)$ to prevent a dangerous s-flower in Maker's graph, where $s=\lfloor q^{1-\delta}\rfloor$, $\delta=\delta_1/4$, $q\ge n^{1/m(G)}$ and n is sufficiently large. ∎

Lemma 7. *For every m-maximal graph G that contains a cycle and every positive $\delta<1$, there exist constants C_2 and n_2 such that for every $n\geq n_2$ and $q\geq C_2 n^{1/m(G)}$ Breaker has a strategy in $\mathbf{G}(G;n,q)$ which does not allows Maker to build $\frac{1}{2}\binom{q}{t}$ simple t-fans, where $t=\lfloor q^{\delta/3}\rfloor$.*

Proof. Let G, δ and t fulfil the assumption of the lemma. We shall show that Maker cannot build $\frac{1}{2}\binom{q}{t}$ simple t-fans if Breaker in $\mathbf{G}(G;n,q)$ uses the strategy stated in Lemma 5 for a $(\mathcal{H},1,q)$-game played on the hypergraph $\mathcal{H}=(V(\mathcal{H}),E(\mathcal{H}))$, where $V(\mathcal{H})=E(K_n)$ and every set $\bigcup_{i=1}^{t} E(F_i^{v_i w_i})$, where $F_1^{v_1w_1},\dots,F_t^{v_tw_t}\subset K_n$ and $\{F_1^{v_1w_1},\dots,F_t^{v_tw_t}\}$ is a simple t-fan, forms an edge of $\mathcal{H}$. (Note that different t-fans can have the same set of edges, i.e., $\mathcal{H}$ may have some multiple edges.)

Indeed, by Lemma 5 at the end of $(\mathcal{H},1,q)$-game Maker's graph contains not more than

$$r=(q+1)f(\mathcal{H},1,q)=(q+1)\sum_{A\in E(\mathcal{H})}(q+1)^{-|A|}$$

edges of $\mathcal{H}$ and by the definition of $\mathcal{H}$ one can estimate r from above by

$$\begin{aligned} r&\leq (q+1)\binom{n}{2}\frac{1}{t!}\left(v(G)!e(G)\binom{n}{v(G)-2}(q+1)^{-e(G)+2}\right)^t\\ &\leq qn^2\left(c_2(n^{1/m(G)}q^{-1})^{e(G)-1}qt^{-1}\right)^t \end{aligned}$$

with a constant c_2 depending on G only. Now we choose constants n_2 and C_2 so that $(c_2^{-1}C_2^{e(G)-1})^{q^{\delta/3}}>2q^{1+2m(G)}$ for $q\geq C_2 n_2^{1/m(G)}$. Then, for $t=\lfloor q^{\delta/3}\rfloor$ and every n such that $n_2\leq n\leq (q/C_2)^{m(G)}$, we have

$$r\leq qn^2(c_2C_2^{1-e(G)}qt^{-1})^t\leq q^{1+2m(G)}(c_2C_2^{1-e(G)})^{q^{\delta/3}}(qt^{-1})^t<\frac{1}{2}\binom{q}{t}.$$

Thus, in the game $\mathbf{G}(G;n,q)$, Breaker can play in such a way that Maker's final graph contains less than $\frac{1}{2}\binom{q}{t}$ simple t-fans. ∎

Fact 8. *For every positive constant $\delta<1$ there exists a constant $q_0=q_0(\delta)$ such that every graph F with $q\geq q_0$ vertices and at most $q^{2-\delta}$ edges contains at least $\frac{1}{2}\binom{q}{t}$ independent sets of size $t=\lfloor q^{\delta/3}\rfloor$.*

Proof. Let $0<\delta<1$, $t=\lfloor q^{\delta/3}\rfloor$, and let F be a graph with $v(F)=q$ and $e(F)=\lfloor q^{2-\delta}\rfloor$. Then the number of subsets of $V(F)$ of t vertices that are not independent obviously does not exceed

$$e(F)\binom{q-2}{t-2}\leq q^{2-\delta}\left(\frac{t}{q}\right)^2\binom{q}{t}<2q^{-\delta/3}\binom{q}{t}<\frac{1}{2}\binom{q}{t}$$

provided q is sufficiently large. ∎

Before the next lemma we make a simple but useful observation. Consider $q \geq q_1 + q_2$ with some natural numbers q, q_1 and q_2. It is obvious that if Breaker in $\mathbf{G}(G;n,q_1)$ plays some strategy that prevents Maker from claiming some structure F_1, then Breaker in $\mathbf{G}(G;n,q)$ can play the same strategy, forgetting about the extra edges he is allowed to choose. Similarly, he can play Breaker's strategy from $\mathbf{G}(G;n,q_2)$, preventing a structure F_2 and, more importantly, he can follow both the strategies simultaneously. In this case, during the game $\mathbf{G}(G;n,q)$, Maker can build copies of neither F_1 nor F_2.

Lemma 9. *For every m-maximal graph G that contains a cycle there exist positive constants C_3 and n_3 such that for every $n \geq n_3$ and $q \geq C_3 n^{1/m(G)}$ Breaker has a strategy such that at no stage of the game $\mathbf{G}(G;n,q)$ Maker's graph contains a dangerous q-fan.*

Proof. Let G be a graph which fulfils the assumption of Lemma 9, constants $\delta < 1$, C_1, c_1 and n_1 be chosen as in Lemma 6, let C_2 and n_2 be as in Lemma 7 and finally let $q_0 = q_0(\delta)$ be defined as in Fact 8. We shall find a winning strategy for Breaker in $\mathbf{G}(G;n,q)$ where $n \geq n_3 = \max(n_1, n_2, (C_3^{-1} q_0)^{m(G)})$, $q \geq C_3 n^{1/m(G)}$ and $C_3 = 2\max(C_1, C_2)$.

Breaker's strategy is based on two strategies, say S_1 and S_2, given by Lemma 6 and Lemma 7 respectively, provided we consider $\mathbf{G}(G;n,q/2)$ in both the cases. Notice that n and $q/2$ are as large as required in the lemmas. Therefore, if Breaker in $\mathbf{G}(G;n,q)$ plays S_1 and S_2 at the same time, then at no point of the game Maker's graph contains a dangerous s-flower with $s = \lfloor (q/2)^{1-\delta} \rfloor$, or $\frac{1}{2}\binom{q/2}{t}$ simple t-fans where $t = \lfloor (q/2)^{\delta/3} \rfloor$.

We claim that then at no moment of the game Maker's graph contains a dangerous q-fans. Indeed, assume that it is not the case and let $\{F_1^{v_1 w_1}, \ldots, F_q^{v_q w_q}\}$ be such a q-fan. Let us define an auxiliary graph F' such that $F_1^{v_1 w_1}, \ldots, F_q^{v_q w_q}$ are vertices of F' and $\{F_i^{v_i w_i}, F_j^{v_j w_j}\}$ is an edge of F' if $|V(F_i^{v_i w_i}) \cap V(F_j^{v_j w_j})| \geq 3$. Observe that F' fulfils the assertions of Fact 8 since, due to Breaker's strategy, Maker's graph contains no dangerous s-flowers which yields $e(F') < (q/2)^{1-\delta} q < q^{2-\delta}$. Moreover we have assumed $q \geq C_3 n^{1/m(G)} \geq q_0$ so, by Fact 8, we have at least $\frac{1}{2}\binom{q}{t}$ independent sets of size t in F' and hence at least $\frac{1}{2}\binom{q}{t}$ simple t-fans in F, which contradicts our assumption. ∎

Proof of Theorem 1(ii). Let G be a graph without isolated vertices such that $v(G) \geq 3$.

We start with the assumption that G is a forest. Then either G contains a path on three vertices, or it consists of $v(G)/2$ disjoint edges. In the first case $m(G)=1$ and, clearly, if $q \geq 2n-2$, then Breaker can win $\mathbf{G}(G;n,q)$ selecting in each move all the edges adjacent to vertices v, v', where $\{v,v'\}$ is the pair chosen by Maker in his last move. In the second case Breaker's winning strategy is even easier — if we set $q \geq n^{1/m(G)} = n^2$ the game is over after the first round.

Now suppose that G contains a cycle. Without loss of generality we can assume that G is m-maximal. Let $C_0 = 2C_3$ and $n_0 = n_3$, where C_3 and n_3 are constants defined as in Lemma 9, and fix $n \geq n_0$, $q \geq C_0 n^{1/m(G)}$. Note that Maker can win $\mathbf{G}(G;n,q)$ only if after some Breaker's move he has a dangerous $\bar{G}$-graph. Thus, after every Maker's move Breaker considers a family $\mathcal{G}$ of all "new" dangerous $\bar{G}$-graphs in Maker's graph. Observe that all $\bar{G}$-graphs from $\mathcal{G}$ contain the last edge selected by Maker and thus they form a dangerous t-fan. Now Breaker can win $\mathbf{G}(G;n,q)$ if he follows his strategy for $\mathbf{G}(G;n,q/2)$ which, according to Lemma 9, allows him to prevent dangerous $(q/2)$-fans in Maker's graph, and uses his remaining $q/2$ choices to block the pairs $\{v_1,w_1\},\dots,\{v_t,w_t\}$ if Maker creates a dangerous t-fan $\{F_t^{v_1w_1},\dots,F_t^{v_tw_t}\}$ with $t < q/2$. ∎

Finally, we remark that the constants c_0 and C_0 in Theorem 1 which follow from the proof are typically far apart, while we expect that they can be chosen arbitrarily close to each other.

Conjecture. *For every graph G and $\varepsilon > 0$ there exist constant $c > 0$ and a natural number n_0 such that for every $n \geq n_0$ the following holds.*

(i) *If $q \leq (c-\varepsilon)n^{1/m(G)}$ then Maker can win the game $\mathbf{G}(G;n,q)$.*
(ii) *If $q \geq (c+\varepsilon)n^{1/m(G)}$ then Breaker has a winning strategy for $\mathbf{G}(G;n,q)$.*

The above conjecture remains open even for very simple cases — in fact for no graph G which contains a cycle the existence of such a constant c for $\mathbf{G}(G;n,q)$ has been shown.

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